PHYSICAL REVIEW VOLUME 136, NUMBER 6A 14 DECEMBER 1964

Scattering of Sound by a Classical Vortex*

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(Received 20 July 1964)

The cross section for the scattering of sound by a vortex is calculated using the linearized equations of classical hydrodynamics. With a circulation κ, the differential cross section in the long-wavelength limit is $\frac{1}{2}\pi(\kappa/2\pi c)^2 k \sin^2\varphi(1-\cos\varphi)^{-2}$, where c is the speed of sound. Possible experimental verification is suggested, with particular reference to liquid He II.

I. INTRODUCTION

EXPERIMENTAL and theoretical studies have shown that rotating liquid He II forms an array of rectilinear vortices.[1–4] In particular, Hall and Vinen[4] found that second sound in rotating He II is strongly damped perpendicular to the axis of rotation. They interpret this as a mutual friction force between the normal fluid and the vortices in the superfluid, proportional to the relative velocity of the two fluids. At low temperatures ($\lesssim 0.5°$K), the normal fluid is composed predominantly of phonons,[5] and the theoretical calculation of the frictional force in that temperature range involves the classical problem of scattering of sound by a vortex.[6]

Nearly a century ago, Kelvin[7] studied the normal oscillation modes of a vortex in an incompressible fluid. The presence of a sound wave, which requires a finite compressibility, greatly complicates the hydrodynamic equations. In the specific case of scattering of sound by a vortex, it is only possible to find an analytic solution in the long-wavelength limit. All the phase shifts in this limit are independent of the core structure of the vortex. The proof of this statement for s waves requires a special treatment, and two standard models are considered, which lead to simple results: (1) a hollow core and (2) a core with uniform vorticity (one that executes solid body rotation. Pitaevskii[6] has previously calculated the cross section for scattering of sound by a vortex, using the Born approximation. For this reason, his results differ from those reported here, which are exact in the long-wavelength limit.

* Research supported in part by the U. S. Air Force Office of Scientific Research, Grant No. AF-AFOSR-130-63.

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[1] L. Onsager, Nuovo Cimento **6**, Suppl. 2, 249 (1949).

[2] R. P. Feynman, *Progress in Low Temperature Physics*, edited by C. J. Gorter (North-Holland Publishing Company, Amsterdam, 1955), Vol. I, p. 17.

[3] W. F. Vinen, Proc. Roy. Soc. (London) **A260**, 218 (1961).

[4] H. E. Hall and W. F. Vinen, Proc. Roy. Soc. (London) **A238**, 204 and 215 (1956).

[5] L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), p. 206.

[6] L. P. Pitaevskii, Zh. Eksperim. i Teor. Fiz. **35**, 1271 (1958) [English transl.: Soviet Phys.—JETP **8**, 888 (1959)].

[7] W. Thomson, *Mathematical and Physical Papers* (Cambridge University Press, New York, 1910), Vol. IV, p. 152.

II. LINEARIZED ACOUSTICAL EQUATIONS

We start with the equations of classical hydrodynamics[8]:

$$\partial\rho/\partial t+\nabla\cdot(\rho\mathbf{v})=0\,, \tag{1}$$

$$\partial\mathbf{v}/\partial t+(\mathbf{v}\cdot\nabla)\mathbf{v}+(1/\rho)\nabla p=0\,, \tag{2}$$

where ρ is the density of the fluid, $\mathbf{v}$ is the velocity field, and p is the pressure. These equations must first of all be rewritten in cylindrical coordinates, (r,φ,z). If (u,v,w) represent the radial, tangential, and axial components of the velocity, Eqs. (1) and (2) become

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(ru\rho)+\frac{1}{r}\frac{\partial}{\partial\varphi}(v\rho)+\frac{\partial}{\partial z}(w\rho)=0\,, \tag{3}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\varphi}+w\frac{\partial u}{\partial z}-\frac{v^2}{r}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0\,, \tag{4}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial\varphi}+w\frac{\partial v}{\partial z}+\frac{uv}{r}+\frac{1}{\rho r}\frac{\partial p}{\partial\varphi}=0\,, \tag{5}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial\varphi}+w\frac{\partial w}{\partial z}+\frac{1}{\rho}\frac{\partial p}{\partial z}=0\,. \tag{6}$$

There is a time-independent solution of the form $u=w=0$, $v=V(r)$, $\rho=D(r)$, $p=P(r)$. In this case, Eqs. (3)–(6) reduce to the single condition that

$$DV^2/r=dP/dr\,, \tag{7}$$

which is merely Bernoulli's theorem. For a vortex with circulation κ, the tangential velocity is

$$V(r)=\kappa/2\pi r\,. \tag{8}$$

We now consider a sound wave superimposed on the steady solution. The components of the velocity, the density, and the pressure become u, $V+v$, w; $D+\rho$; $P+p$, where u, v, w; ρ; p are the oscillating components. In the approximation of linear acoustics,[9] terms quadratic in the sound field are neglected. The resulting

[8] H. Lamb, *Hydrodynamics* (Dover Publications, New York, 1945), 6th ed., Chap. 1, pp. 4, 6.

[9] P. M. Morse and K. U. Ingard, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1961), Vol. XI/1. p. 5.

equations are

$$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(ruD)+\frac{1}{r}\frac{\partial}{\partial\varphi}(V\rho+vD)+\frac{\partial}{\partial z}(wD)=0, \quad (9)$$

$$\frac{\partial u}{\partial t}+\frac{V}{r}\frac{\partial u}{\partial\varphi}-\frac{2Vv}{r}+\frac{1}{D}\frac{\partial p}{\partial r}-\frac{\rho}{D}\frac{V^2}{r}=0, \quad (10)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial V}{\partial r}+\frac{V}{r}\frac{\partial v}{\partial\varphi}+\frac{uV}{r}+\frac{1}{Dr}\frac{\partial p}{\partial\varphi}=0, \quad (11)$$

$$\frac{\partial w}{\partial t}+\frac{V}{r}\frac{\partial w}{\partial\varphi}+\frac{1}{D}\frac{\partial p}{\partial z}=0, \quad (12)$$

which are not restricted to the velocity field for a vortex. At sound frequencies, small changes in the pressure and density are related by

$$p=c^2\rho, \quad (13)$$

where c is the speed of sound. Although c is not strictly independent of position, it is taken to be constant in this calculation. It can be shown (at least for a vortex in an ideal gas) that this assumption introduces no error in lowest order.

Using (9)–(12) it is a simple matter to verify that a sound wave may propagate unchanged along the axis of the vortex. There is no scattering in this case, and we shall therefore specialize to motion in the plane perpendicular to the axis of the vortex ($w=0$). Harmonic time dependence ($e^{-i\omega t}$) is appropriate for a sound wave, and it is also useful to expand in a Fourier series:

$$\rho(r,\varphi)=\frac{1}{2\pi}\sum_{\ell=-\infty}^{\infty}e^{i\ell\varphi}\rho_\ell(r), \quad (14)$$

with similar expansions for u and v. By the use of (13) and (14), the equations of linear acoustics may be simplified to the following set:

$$\left(-i\omega+i\frac{\ell V}{r}\right)\rho_\ell+\frac{1}{r}\frac{d}{dr}(u_\ell rD)+\frac{i\ell D}{r}v_\ell=0, \quad (15)$$

$$\left(-i\omega+i\frac{\ell V}{r}\right)u_\ell-\frac{2Vv_\ell}{r}+\frac{c^2}{D}\frac{d\rho_\ell}{dr}-\frac{V^2}{rD}\rho_\ell=0, \quad (16)$$

$$\left(-i\omega+i\frac{\ell V}{r}\right)v_\ell+u_\ell\left(\frac{dV}{dr}+\frac{V}{r}\right)+\frac{ic^2\ell}{Dr}\rho_\ell=0. \quad (17)$$

The last equation is an algebraic one, and it may be used to eliminate v_ℓ from the other two equations. The result is a pair of coupled, linear, first-order differential equations for u_ℓ and ρ_ℓ. A good bit of manipulation yields the following pair:

$$\frac{d^2\rho_\ell}{dr^2}+\frac{1}{r}\frac{d\rho_\ell}{dr}+\left(\frac{\omega^2}{c^2}-\frac{\ell^2}{r^2}\right)\rho_\ell-s_\ell\frac{d\rho_\ell}{dr}+t_\ell\rho_\ell=0, \quad (18)$$

$$u_\ell=a_\ell\frac{d\rho_\ell}{dr}+b_\ell\rho_\ell, \quad (19)$$

where

$$s_\ell=\frac{V^2}{rc^2}+\frac{d}{dr}\ln A_\ell, \quad (20)$$

$$t_\ell=-\frac{2\ell V\omega}{rc^2}+\left(\frac{\ell V}{rc}\right)^2-\frac{2V}{rc^2}\left(\frac{V}{r}+2\frac{dV}{dr}\right)-\frac{b_\ell}{a_\ell}\frac{d}{dr}\ln A_\ell$$
$$+2\ell\left(\omega-\frac{\ell V}{r}\right)^{-1}\left(\frac{V}{r^3}-\frac{1}{r^2}\frac{dV}{dr}-\frac{V^3}{r^3c^2}\right), \quad (21)$$

$$A_\ell=i\left(\omega-\frac{\ell V}{r}\right)^2-\frac{2iV}{r^2}\frac{d}{dr}(rV), \quad (22)$$

$$a_\ell=c^2(DA_\ell)^{-1}(\omega-\ell V/r), \quad (23)$$

$$b_\ell=-(r^2DA_\ell)^{-1}[V^2r(\omega-\ell V/r)+2V\ell c^2]. \quad (24)$$

For the vortical velocity field (8), s and t vanish for large r, and the solutions of (18) behave asymptotically like a linear combination of Bessel functions. It is convenient to introduce the concept of phase shifts, exactly as in three-dimensional scattering. Since the formulas are slightly different in two dimensions, the relevant results are derived in the next section.

III. PHASE SHIFTS AND SCATTERING IN TWO DIMENSIONS

The problem of interest is the scattering of an incident plane wave; far from the vortex ($r\to\infty$) we assume that

$$\rho(r,\varphi)\to e^{i\mathbf{k}\cdot\mathbf{r}}+(\tfrac{1}{2}\pi k)^{1/2}f(\varphi)H_0^{(1)}(kr)$$
$$=e^{i\mathbf{k}\cdot\mathbf{r}}+r^{-1/2}f(\varphi)\exp[i(kr-\tfrac{1}{4}\pi)], \quad (25)$$

where $H_0^{(1)}$ is the Hankel function[10] appropriate for outgoing waves. The procedure is exactly as in the three-dimensional scattering problem.[11] The solution of (18) that satisfies the inner boundary condition must behave for large r like

$$\rho_\ell(r)\to(2/\pi kr)^{1/2}\cos[kr-\tfrac{1}{2}(\ell+\tfrac{1}{2})\pi+\delta_\ell]. \quad (26)$$

A comparison of (25) and (26) yields the following relation between the scattering amplitude $f(\varphi)$ and the phase shifts:

$$f(\varphi)=(2\pi k)^{-1/2}\sum_\ell[\exp(2i\delta_\ell)-1]e^{i\ell\varphi}. \quad (27)$$

Hence, the scattered density for large r is of the form

$$\rho_{sc}(r,\varphi)\to r^{-1/2}f(\varphi)\exp[i(kr-\tfrac{1}{4}\pi)]. \quad (28)$$

Equation (19) gives the radial component of velocity in terms of ρ_ℓ, and, for $r\to\infty$, we have

$$D\omega u_\ell=-ic^2(d\rho_\ell/dr). \quad (29)$$

[10] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1962), 2nd ed., pp. 73, 197.

[11] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., p. 103.

In this limit, the summation over ℓ is trivial, and the result is

$$u_{sc}(r,\varphi)\to(D\omega)^{-1}c^2kr^{-1/2}f(\varphi)\exp[i(kr-\tfrac{1}{4}\pi)]. \quad (30)$$

The cross section is defined in terms of the energy flux through a cylinder at infinity. The problem of acoustic energy flux in a moving medium is a little tricky,[12,13] and the discussion is simplified by noting that at large distances, any small element of a cylinder is essentially a plane. In this limit we may use a result derived by Ribner[13] for the energy flux J perpendicular to a uniform velocity field:

$$J=\langle pu\rangle_{\text{av}}+DV\langle uv\rangle_{\text{av}}, \quad (31)$$

where $\langle\cdots\rangle_{\text{av}}$ means the average over one cycle. The second term describes the effect of the steady velocity field. All the oscillating quantities behave like $r^{-1/2}$ for large r, and the second term of (31) is therefore of order r^{-1} with respect to the first. The net result is just what one would write down for a stationary medium.

The outward energy flow per unit time into an angular element $d\varphi$ of a cylinder of unit length is

$$\langle pu\rangle_{\text{av}}rd\varphi=\tfrac{1}{2}\text{Re}(p_{sc}u_{sc}^*)rd\varphi. \quad (32)$$

With the use of (13), (28), and (30), Eq. (32) becomes

$$\tfrac{1}{2}(D\omega)^{-1}c^4k|f(\varphi)|^2d\varphi. \quad (33)$$

For the incident plane wave,

$$\rho_{\text{inc}}=e^{i\mathbf{k}\cdot\mathbf{r}},$$

and the incident flux is

$$\tfrac{1}{2}(D\omega)^{-1}c^4k.$$

The cross section, which is defined as the energy scattered per unit time per unit incident energy flux into a unit angular interval, is given by

$$\sigma(\varphi)=|f(\varphi)|^2. \quad (34)$$

The total cross section is

$$\sigma=\int_0^{2\pi}\sigma(\varphi)d\varphi=4k^{-1}\textstyle\sum_\ell \sin^2\delta_\ell. \quad (35)$$

Notice that (in two dimensions) these have the dimension of a length, not an area.

IV. THE S-WAVE PHASE SHIFT

In a general two-dimensional scattering problem, the s-wave phase shift behaves like $(\ln k)^{-1}$ for small k. This is easily seen by the following argument[14]: We consider scattering by a perturbation $U(r)$, where U is negligible for $r>r_0$. Then the wave equation for any physical quantity χ is

$$\chi''(r)+r^{-1}\chi'(r)+[k^2-U(r)]\chi(r)=0. \quad (36)$$

In the region $r\ll k^{-1}$, the term in k^2 is unimportant and the equation becomes

$$\chi''(r)+r^{-1}\chi'(r)-U(r)\chi(r)=0. \quad (37)$$

For sufficiently large r, U may also be neglected, so that the solution for $r_0\ll r\ll k^{-1}$ is

$$\chi(r)=c_1+c_2\ln r. \quad (38)$$

On the other hand, U is negligible for $r_0\ll r$, in which case Eq. (36) reduces to

$$\chi''(r)+r^{-1}\chi'(r)+k^2\chi(r)=0; \quad (39)$$

the general solution of (39) is

$$\chi(r)=AJ_0(kr)+BY_0(kr). \quad (40)$$

Equations (38) and (40) are both valid for $r_0\ll r$, so that the constants A and B are determined:

$$A=c_1-c_2(\gamma+\ln\tfrac{1}{2}k),$$
$$B=\tfrac{1}{2}\pi c_2, \quad (41)$$

where γ is Euler's constant, 0.577. When $kr\gg1$, the asymptotic behavior of the Bessel functions can be used,[15] and comparison of (40) with (26) yields

$$\tfrac{1}{2}\pi\cot\delta_0=\ln\tfrac{1}{2}k+\gamma-(c_1/c_2). \quad (42)$$

Thus for small k, the s-wave phase shift behaves like $(\ln k)^{-1}$.

The cross section calculated from (42) diverges like $k^{-1}(\ln k)^{-2}$ for long wavelengths. Such behavior occurs in many physical situations. The quantum-mechanical scattering by a hard cylinder[16] and the transmission of sound through an aperture in a screen bounded by parallel straight edges[17] are typical examples. There is, however, one well-known exception: The cross section for scattering of sound by a rigid cylinder is proportional to k^3 for small k.[18] This difference is due to the inner boundary conditions on χ.[16] In the quantum-mechanical scattering problem, χ is required to vanish at the surface of the cylinder, while in the acoustic problem, the gradient of χ vanishes. A detailed analysis shows that for the latter case, the "constant" c_2 in (38) is in fact a small quantity of order k^2. The s-wave phase shift in (42) is then also proportional to k^2, which explains the apparent contradiction between the general form of the

[12] P. M. Morse and K. U. Ingard, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1961), Vol. XI/1, p. 9.
[13] H. S. Ribner, J. Acoust. Soc. Am. **29**, 435 (1957).
[14] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), p. 403.
[15] G. N. Watson, *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, New York, 1962), 2nd ed., p. 199.
[16] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Part II, p. 1382.
[17] H. Lamb, *Hydrodynamics* (Dover Publications, New York, 1945), 6th ed., p. 533. See also J. W. Strutt, Baron Rayleigh, *Scientific Papers* (Cambridge University Press, New York, 1903), Vol. IV, p. 283.
[18] J. W. Strutt, Baron Rayleigh, *The Theory of Sound* (MacMillan and Company, Ltd., London, 1929), Vol. II, p. 309.

cross section and Rayleigh's result for the acoustic problem.[18]

A similar situation occurs when sound is scattered by a vortex. If ℓ is zero, Eq. (18) simplifies considerably, and, with the vortical velocity field (8), we find

$$\rho_0''(x)+x^{-3}(x^2-1)\rho_0'(x)+(\nu^2+2x^{-4})\rho_0(x)=0, \quad (43)$$

where $x=2\pi cr/\kappa$ and $\nu=\kappa k/2\pi c\ll 1$. For the region $x^2\ll\nu^{-1}$, Eq. (43) reduces to

$$\rho_0''(x)+x^{-3}(x^2-1)\rho_0'(x)+2x^{-4}\rho_0(x)=0. \quad (44)$$

This may be solved with the substitution

$$\rho_0(x)=y_0(x)\exp(-\tfrac{1}{2}x^{-2}), \quad (45)$$

which leads to

$$y_0''(x)+x^{-3}(x^2+1)y_0'(x)=0. \quad (46)$$

The general solution of (46) is

$$y_0(x)=A+B\int dx x^{-1}\exp(\tfrac{1}{2}x^{-2}), \quad (47)$$

where A and B are constants. A simple change of variable yields

$$\rho_0(x)=A\exp(-\tfrac{1}{2}x^{-2})+B\exp(-\tfrac{1}{2}x^{-2})\mathrm{Ei}(\tfrac{1}{2}x^{-2}). \quad (48)$$

The exponential integral in (48) is defined by

$$\mathrm{Ei}(x)=\int_{-\infty}^{x}dt t^{-1}e^t, \quad (49)$$

which is interpreted as a principal value for $x>0$.[19]

The ratio of the constants in (48) is determined by the boundary conditions. The two chosen models of the vortex core lead to especially simple expressions, and, in fact, give the same behavior in the long-wavelength limit. If the vortex has a hollow core of radius a, then the oscillating component of the pressure must vanish at the inner edge of the vortex. We consider a particle whose radial position in the absence of a sound wave is r_0.[7] Its position in the presence of the sound wave is given by

$$r=r_0+\int dt u e^{-i\omega t}=r_0+iu\omega^{-1}e^{-i\omega t}. \quad (50)$$

The corresponding change in the pressure is obtained from (7) and is

$$\Delta P=r_0^{-1}DV^2(r-r_0). \quad (51)$$

The oscillating component of the pressure must vanish at $r=a$, and it follows that

$$\Delta P(a)+p(a)=\Delta P(a)+c^2\rho(a)=0. \quad (52)$$

When (52) is combined with (50) and (51), the correct boundary condition for a vortex with a hollow core is

$$iDV^2(\omega a)^{-1}u(a)+c^2\rho(a)=0. \quad (53)$$

Equation (19) gives the radial velocity in terms of ρ, and for $\ell=0$, Eq. (53) becomes

$$a\rho_0'(a)-(V/c)^2\rho_0(a)=-(\omega a/V)^2\rho_0(a). \quad (54)$$

For long wavelengths, the left member of (54) is a small quantity of order $(ka)^2$.

The second model considered is a core with uniform vorticity, in which the core rotates like a solid body. The tangential velocity of the vortex is given by

$$\begin{aligned} V(r)&=\kappa/2\pi r \qquad r>a\\ &=\kappa r/2\pi a^2 \quad r<a, \end{aligned} \quad (55)$$

and the boundary conditions require the continuity of the pressure and of the radial velocity at $r=a$. Equation (19) for the radial velocity contains the factor

$$A_0=i[\omega^2-(2V/r^2)(d/dr)(rV)], \quad (56)$$

which is discontinuous at $r=a$. The condition on the radial velocity can be reduced to the following equation:

$$\begin{aligned} &a\rho_>'(a)-(V/c)^2\rho_>(a)\\ &=\omega^2[\omega^2-(\kappa/\pi a^2)^2]^{-1}[a\rho_<'(a)-(V/c)^2\rho_<(a)], \end{aligned} \quad (57)$$

where $\rho_>$ and $\rho_<$ refer to the values for r greater or less than a. In the long-wavelength limit, the left member of (57) is of order $(ka)^2$, which is equivalent to the boundary condition (54) for a hollow core. Furthermore, the quantity on the left side of both equations (54) and (57) is proportional to $u(a)$, so that the radial velocity at the boundary vanishes like k^2 in both models. This is to be compared with the scattering of sound by a rigid cylinder, where $u(a)$ vanishes. For small k, the s-wave scattering by a vortex and by a rigid cylinder are essentially the same.

Both (54) and (57) can be written in dimensionless form as

$$x_0\rho_0'(x_0)-x_0^{-2}\rho_0(x_0)=O(\nu^2), \quad (58)$$

where $x_0=2\pi ca/\kappa$. When this boundary condition is applied to (48), the ratio B/A is found to be of order ν^2. The phase shift is determined from the behavior of (48) for large x, which is

$$\rho_0(x)\to A+B(\gamma-\ln 2x^2), \quad (59)$$

where γ is Euler's constant.[19] The coefficient of the logarithmic term is of order ν^2, which shows that the s-wave phase shift is also of order ν^2. This result is not restricted to the simple cases considered above; any model for which $u(a)=O(k^2)$ leads to an s-wave phase shift of order k^2.

V. HIGHER PARTIAL WAVES

The scattering in the higher angular momentum states ($\ell\neq 0$) will now be considered. With the same

[19] There are several different definitions of the exponential integral. We follow the recommendation of A. Fletcher, J. C. P. Miller, L. Rosenhead, and L. J. Comrie, *An Index of Mathematical Tables* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1962), 2nd ed., Vol. I, pp. 267, 268.

dimensionless variables as in (43), Eqs. (18), (20), and (21) become

$$\rho_\ell''+x^{-1}\rho_\ell'+(\nu^2-\ell^2x^{-2})\rho_\ell-s_\ell\rho_\ell'+t_\ell\rho_\ell=0\,, \qquad (60)$$

$$s_\ell=x^{-3}+4\ell x^{-1}(\nu x^2-\ell)^{-1}\,, \qquad (61)$$

$$t_\ell=-2\ell\nu x^{-2}+(\ell^2+2)x^{-4}+2\ell x^{-4}(\nu x^2-\ell)^{-1}(1+2x^2) +8\ell^2x^{-2}(\nu x^2-\ell)^{-2}. \qquad (62)$$

Both $x=0$ and $x=\infty$ are irregular singular points of (60), while

$$x=\pm(\ell/\nu)^{1/2}$$

are regular singular points. It is remarkable that the regular singularities can be removed by the substitution

$$\rho_\ell(x)=y_\ell(x)\exp(\tfrac{1}{2}\textstyle\int s_\ell)\,. \qquad (63)$$

For large x, the ratio ρ_ℓ/y_ℓ approaches unity, so that ρ_ℓ and y_ℓ have the same asymptotic phase. The resulting equation for y_ℓ is

$$y_\ell''+x^{-1}y_\ell'+y_\ell[\nu^2-\lambda^2x^{-2}+U(x)]=0\,, \qquad (64)$$

where

$$U(x)=(\ell^2+1)x^{-4}-\tfrac{1}{4}x^{-6}\,, \qquad (65)$$

$$\lambda^2=\ell^2+2\ell\nu\,. \qquad (66)$$

The only singular points of (64) are $x=0$ and $x=\infty$, and the equation is of a form familiar in scattering theory.

If the "potential" $U(x)$ behaves like x^{-n} for large x, it is easy to show that the phase shift due to U behaves for small ν like $\nu^{2\lambda}$ if $2\lambda<n-2$ and like ν^{n-2} if $2\lambda>n-2$. Landau and Lifshitz[14] derive the corresponding conditions for three-dimensional scattering. With the specific form of U given in (65), the asymptotic exponent is $n=4$. It follows immediately from (66) that the part of the phase shift caused by U is of order ν^2 for small ν. This holds even for $\ell=-1$, where the phase shift behaves like

$$\nu^2\exp(-2\nu\ln\nu)\,,$$

because

$$|\exp(-\nu\ln\nu)|<\exp(e^{-1})$$

for

$$0\leq\nu<e^{-1}.$$

It is therefore sufficient to examine the phase shift due to the explicit appearance of ν in (66). In the absence of the vortex, the correct solution of (64) is $J_\ell(\nu x)$, whose asymptotic phase is

$$-\tfrac{1}{2}\pi(|\ell|+\tfrac{1}{2}).$$

The "potential" U may be neglected in (64) since its contribution to the phase shift is of higher order. Hence the correct solution in the presence of the vortex is

$$J_{|\lambda|}(\nu x)\,,$$

whose asymptotic phase is

$$-\tfrac{1}{2}\pi[(\ell^2+2\ell\nu)^{1/2}+\tfrac{1}{2}].$$

The phase shift is the difference between the two asymptotic phases:

$$\delta_\ell=-\tfrac{1}{2}\pi[(\ell^2+2\ell\nu)^{1/2}-|\ell|]\approx-\tfrac{1}{2}\pi\nu\,\mathrm{sgn}\ell\,, \qquad (67)$$

where

$$\mathrm{sgn}\ell=\ell/|\ell|\,.$$

This is of order ν, so that the other effects (of order ν^2) are in fact negligible.

VI. THE SCATTERING CROSS SECTION

The scattering amplitude is now easily computed from (67) and (27). In the long-wavelength limit, the phase shifts (67) are small. It is sufficient to approximate

$$\exp(2i\delta_\ell)-1$$

by

$$-\pi i\nu\,\mathrm{sgn}\ell\,,$$

so that

$$f(\varphi)=-\pi i\nu(2\pi k)^{-1/2}\textstyle\sum' e^{i\ell\varphi}\,\mathrm{sgn}\ell\,, \qquad (68)$$

where the prime means: omit the term with $\ell=0$. The summation in (68) can be evaluated as follows:

$$\begin{aligned}\textstyle\sum' e^{i\ell\varphi}\,\mathrm{sgn}\ell&=\textstyle\sum_{\ell>0}(e^{i\ell\varphi}-e^{-i\ell\varphi})\\ &=2i\,\mathrm{Im}\textstyle\sum_{\ell>0}e^{i\ell\varphi}=2i\,\mathrm{Im}(1-e^{i\varphi})^{-1}\\ &=i\sin\varphi(1-\cos\varphi)^{-1}.\end{aligned} \qquad (69)$$

The scattering amplitude is given by

$$f(\varphi)=\nu(\pi/2k)^{1/2}\sin\varphi(1-\cos\varphi)^{-1}\,, \qquad (70)$$

with the corresponding differential cross section:

$$\sigma(\varphi)=|f(\varphi)|^2=\tfrac{1}{2}\pi(\kappa/2\pi c)^2k\sin^2\varphi(1-\cos\varphi)^{-2}. \qquad (71)$$

The total cross section computed from (71) diverges, which reflects the long-range nature of the vortical velocity field, in a manner similar to the Rutherford cross section. In any attenuation experiment, however, there is a natural cutoff φ_0, which is essentially the angle subtended by the detector. A wave scattered through an angle $\varphi<\varphi_0$ is measured as part of the unscattered wave and thus fails to contribute to the total cross section. The experimental quantity of interest is

$$\sigma_T(\varphi_0)=\int_{\varphi_0}^{2\pi-\varphi_0}\sigma(\varphi)d\varphi\,, \qquad (72)$$

and from (71) we find

$$\sigma_T(\varphi_0)=\pi(\kappa/2\pi c)^2k[2\cot(\tfrac{1}{2}\varphi_0)+\varphi_0-\pi]. \qquad (73)$$

Chase, Fineman, and Millett[20] have studied the attenuation of sound in rotating liquid He II, where the circulation is h/m, in which m is the mass of the He atom. Rotation speeds of up to 20 rad/sec produced no measurable effect at a frequency of 10^6 cps, and they concluded that the attenuation due to the vortices was less than 10^{-3} cm^{-1}. If the cutoff is taken as 10^{-2} rad,

[20] C. E. Chase, J. Fineman, and W. E. Millett, Physica **25**, 631 (1959).

the attenuation computed from (73) is 2.9×10^{-8} cm^{-1}, which is well within the experimental limit.

In many physical processes, the relevant quantity is not the total cross section, but rather the transport cross section, defined by

$$\sigma^*=\int\sigma(\varphi)(1-\cos\varphi)d\varphi. \tag{74}$$

A typical example is the transfer of momentum from a sound wave to a vortex, for which σ^* is finite in spite of the long range of the velocity field. From (71), it is straightforward to find that

$$\sigma^*=\tfrac{1}{4}(\kappa/c)^2k=\pi^2(\hbar/mc)^2k, \tag{75}$$

where the last result is valid for liquid He II ($\kappa=h/m$). The transport cross section (75) is identical with Pitaevskii's[6] quoted result, but the agreement is perhaps fortuitous; a factor of $\frac{1}{2}$ appears to be missing in each of his equations (26) and (29). The differential cross section (71) differs from his calculation, but the origin of the contradiction is not immediately apparent in his work. It is more obvious in a treatment using partial waves. Pitaevskii works throughout with two-dimensional partial differential equations and applies the Born approximation to the unseparated equation corresponding to (60). In addition to the first term of (62), he was thus forced to include two terms of order $(\nu x^4)^{-1}$. These latter terms cancel in the exact equation (64), and their contribution to the scattering amplitude is spurious. Furthermore, Pitaevskii neglects from the start all terms quadratic in the velocity field of the vortex, as a result of which the s-wave scattering is entirely omitted. This approximation must be justified by a detailed examination of specific models, as in Sec. IV.

VII. DISCUSSION

The only way to obtain a measurable scattering cross section in liquid He II is to increase the frequency of the sound wave. The extremely small magnitude of the cross sections, however, probably precludes the use of an externally generated sound wave to verify (71) or (75). A more feasible method is based on the excitations in liquid He II, which, below 0.5°K, are almost entirely phonons of relatively high frequency. As an example, the temperature 0.4°K corresponds to a frequency of 8.3×10^9 cps, for which the transport cross section (75) is 0.095 Å (Chase *et al.* worked at 10^6 cps). Pitaevskii[6] shows that the force exerted on a vortex by the gas of phonons is proportional to T^5, so that the measurement of σ^* would be most easily carried out in the temperature range from 0.2 to 0.5°K. At temperatures above 0.6°K, the phonon component of the normal fluid is negligible. Hence, Hall and Vinen,[4] working above 1.2°K, observed essentially pure proton scattering by a vortex. Rayfield and Reif[21] have recently studied the energy loss of a large vortex ring over the range $0.28°\text{K}\leqq T\leqq0.7°\text{K}$; they conclude that the measurements are "not inconsistent" with Eq. (75), the theoretical value for the phonon transport cross section.

It is interesting to examine the physical basis for the scattering of sound by a vortex. The usual wave equation is no longer valid in a moving medium, and the basic equation of linear acoustics[9] is

$$c^{-2}(\partial/\partial t+\mathbf{V}\cdot\nabla)^2\rho-\nabla^2\rho=0. \tag{76}$$

If a vortex is the source of the velocity field, $\mathbf{V}\cdot\nabla$ reduces to

$$(V/r)(\partial/\partial\varphi),$$

which accounts for the appearance of the combination

$$(\omega-\ell V/r)$$

in (15)–(17) instead of ω. The only scattering in the long-wavelength limit is due to the first term of (21), which is merely the middle term of

$$(\omega-\ell V/r)^2.$$

In fact, if only this term is retained, a calculation of the phase shift in Born approximation yields the same result as (67). The quantum mechanical treatment of a moving medium is wholly analogous to the classical one. The energy E is replaced by

$$(E-\mathbf{V}\cdot\mathbf{p}),$$

and if the velocity of the medium is small, the second term can be regarded as a perturbation.[4]

The inner boundary conditions are not critical in the present calculation. They function primarily to reduce the s-wave phase shift from a dominant role to a negligible one but play no part at all in the higher partial waves. Any model in which δ_0 is of order k^2 gives the same differential cross section to leading order. Such models differ only in higher (probably unmeasurable) corrections. It is, of course, not surprising that a long-wavelength sound wave can detect only the long-range velocity field of the vortex. It appears that no existing experimental techniques can probe the vortex core. At present, however, there is no theory of core structure to be verified.

ACKNOWLEDGMENTS

I should like to thank Professor P. C. Martin, whose comments provided the impetus for this calculation. Correspondence with Dr. C. E. Chase and discussions with Professor F. Reif and Professor K. M. Watson were important in the final formulation of the results, for which I am most grateful.

[21] G. W. Rayfield and F. Reif, Phys. Rev. **136**, A1194 (1964).